

ON FRAMED QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. A noncommutative-geometric formalism of framed principal bundles is sketched, in a special case of quantum bundles (over quantum spaces) possessing classical structure groups. Quantum counterparts of torsion operators and Levi-Civita type connections are analyzed. A construction of a natural differential calculus on framed bundles is described. Illustrative examples are presented.

1. INTRODUCTION

In classical differential geometry the formalism of principal bundles plays a central role. In particular, a distinguished role is played by *framed* principal bundles, characterized as covering bundles of subbundles of the bundle of linear frames of the base manifold. Framed principal bundles provide a natural conceptual framework for the study of fundamental classical differential-geometric structures.

In this paper classical idea of a framed principal bundle will be incorporated into an appropriate noncommutative-geometric [1] context. The main entities figuring in the game will be quantum principal bundles (over quantum spaces) possessing (compact) classical structure groups.

Conceptually, the paper is based on a general theory of quantum principal bundles, presented in [2].

The paper is organized as follows.

Section II is devoted to the definition and general properties of frame structures. As first, a very important class of integrable frame structures will be defined. Roughly speaking, integrable frame structures naturally induce “Levi-Civita” type connections on the bundle. On the other hand, such connections are compatible, in the appropriate sense, with internal geometrical structure of the bundle. This fact opens a possibility to construct, in a fully intrinsic manner, the complete differential calculus on the bundle, starting from a given integrable frame structure.

More precisely, starting from a quantum principal bundle P and an appropriate representation u of the structure group G it is possible to construct a graded $*$ -algebra \mathfrak{hor}_P playing the role of horizontal forms on P (together with the right (co)action of G on this algebra). Then, the graded $*$ -algebra Ω_M representing differential forms on the base manifold can be described as the subalgebra of \mathfrak{hor}_P consisting of G -invariant elements. On the other hand, if an integrable frame structure on P (with respect to u) is given, then it is possible to define, in a natural manner, a differential on the algebra Ω_M . Finally, starting from \mathfrak{hor}_P and Ω_M and applying ideas of [3] it is possible to construct (via the concept of a preconnection)

the whole graded differential $*$ -algebra Ω_P representing differential forms on the bundle P .

In the framework of (integrable) frame structures, it is possible to define torsion operators associated to connection forms (as in the classical theory). These operators are analyzed at the end of Section II.

It is important to mention that connection forms will appear implicitly, represented by the corresponding covariant derivatives. Further, all connections appearing in this study will be regular and multiplicative (in the sense of [2])

In Section III some examples of framed quantum principal bundles are presented.

Finally, in Section IV concluding remarks are made.

2. FRAME STRUCTURES

Let M be a quantum space, represented by a $*$ -algebra \mathcal{V} . Let G be an ordinary compact (matrix) Lie group. Concerning group entities, the notation of [6] will be followed (although in the classical context). In particular the group G will be described by a (commutative) Hopf $*$ -algebra \mathcal{A} consisting of polynomial functions on G . The group structure is encoded in the coproduct $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ and the antipode $\kappa: \mathcal{A} \rightarrow \mathcal{A}$.

Let u be a real unitary representation of G in the standard n -dimensional unitary space \mathbb{C}^n . We shall assume that the kernel of u is discrete. Furthermore, u will be interpreted as a right comodule structure map $u: \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathcal{A}$, so that

$$u(e_i) = \sum_{j=1}^n e_j \otimes u_{ji},$$

where u_{ij} are matrix elements of u (and e_i are absolute basis vectors in \mathbb{C}^n).

Let $P = (\mathcal{B}, i, F)$ be a quantum principal G -bundle over M . Here, \mathcal{B} is a (unital) $*$ -algebra consisting of appropriate “functions” on the bundle, $i: \mathcal{V} \rightarrow \mathcal{B}$ is the dualized “projection” of P on M , and $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is the dualized “right action” of G on P . The elements of \mathcal{V} will be identified with their images from $i(\mathcal{V})$. The algebra \mathcal{B} is understandable as a bimodule over \mathcal{V} , in a natural manner.

Let us assume that a system $\tau = (\partial_1, \dots, \partial_n)$ of \mathcal{B} -valued hermitian derivations $\partial_i: \mathcal{V} \rightarrow \mathcal{B}$ is given such that

$$(1) \quad F\partial_i(f) = \sum_{j=1}^n \partial_j(f) \otimes u_{ji}$$

for each $i \in \{1, \dots, n\}$ and $f \in \mathcal{V}$. Finally, let us assume that the following *completeness condition* holds.

There exist a natural number d and elements $b_{i\alpha} \in \mathcal{B}$ and $v_{i\alpha} \in \mathcal{V}$ (where $\alpha \in \{1, \dots, d\}$ and $i \in \{1, \dots, n\}$) such that

$$(2) \quad \sum_{\alpha} b_{i\alpha} \partial_j(v_{i\alpha}) = \delta_{ij} 1,$$

for each $i, j \in \{1, \dots, n\}$.

Definition 2.1. Every system τ satisfying the above conditions is called a *frame structure* on P (relative to u).

Definition 2.2. A frame structure τ is called *integrable* iff there exists a system $\hat{\tau} = (X_1, \dots, X_n)$ of hermitian derivations $X_i: \mathcal{B} \rightarrow \mathcal{B}$ satisfying

$$(3) \quad FX_j = \sum_{k=1}^n (X_k \otimes u_{kj})F$$

$$(4) \quad X_i \lrcorner \mathcal{V} = \partial_i$$

$$(5) \quad X_i \partial_j - X_j \partial_i = 0$$

for each $i, j \in \{1, \dots, n\}$.

In the following, it will be assumed that the bundle P is endowed with a fixed integrable frame structure τ . Let us consider a graded $*$ -algebra

$$\mathfrak{hor}_P = \mathcal{B} \otimes \mathbb{C}_n^\wedge,$$

where \mathbb{C}_n^\wedge is the corresponding external $*$ -algebra. The elements of \mathfrak{hor}_P will be interpreted as “horizontal forms” on P . Algebras \mathcal{B} and \mathbb{C}_n^\wedge are naturally understandable as subalgebras of \mathfrak{hor}_P . We shall denote by $\theta_i \leftrightarrow 1 \otimes e_i$ special horizontal 1-forms corresponding to absolute basis vectors. Let $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ be the product of actions F and u^\wedge , where $u^\wedge: \mathbb{C}_n^\wedge \rightarrow \mathbb{C}_n^\wedge \otimes \mathcal{A}$ is the representation of G in \mathbb{C}_n^\wedge , induced by u .

To each extension $\hat{\tau} = (X_1, \dots, X_n)$ of τ it is possible to associate a first-order antiderivation $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ such that

$$(6) \quad \nabla(b) = \sum_{k=1}^n X_k(b) \theta_k$$

$$(7) \quad \nabla(\theta_i) = 0$$

for each $b \in \mathcal{B}$ and $i \in \{1, \dots, n\}$. Moreover,

$$(8) \quad \nabla* = *\nabla$$

$$(9) \quad F^\wedge \nabla = (\nabla \otimes \text{id})F^\wedge.$$

Let $\mathfrak{fr}(P)$ be the set of all antiderivations ∇ constructed in this way (\Leftrightarrow the set of all “frame extensions” $\hat{\tau}$). Clearly, $\mathfrak{fr}(P)$ is a real affine space, in a natural manner.

Let $\Omega_M \subseteq \mathfrak{hor}_P$ be a graded $*$ -subalgebra consisting of all F^\wedge -invariant elements. Clearly, $\Omega_M^0 = \mathcal{V}$. Elements of Ω_M will play the role of differential forms on M . Because of (9) we have

$$\nabla(\Omega_M) \subseteq \Omega_M,$$

for each $\nabla \in \mathfrak{fr}(P)$.

Lemma 2.1. (i) *There exists the common restriction $d_M: \Omega_M \rightarrow \Omega_M$ of all maps $\nabla \in \mathfrak{fr}(P)$.*

(ii) *The space Ω_M is linearly spanned by elements of the form*

$$(10) \quad w = f_0 d_M f_1 \dots d_M f_k$$

where $f_i \in \mathcal{V}$.

(iii) *We have*

$$(11) \quad d_M^2 = 0.$$

Proof. Let us fix $\nabla \in \mathfrak{fr}(P)$ and let $d_M = \nabla \lrcorner \Omega_M$. The completeness condition implies that each element $w \in \mathfrak{hor}_P^k$ can be written in the form

$$w = \sum_i w_i d_M f_i$$

where $f_i \in \mathcal{V}$ and $w_i \in \mathfrak{hor}_P^{k-1}$. This follows from the equality

$$(12) \quad \theta_i = \sum_{\alpha} b_{i\alpha} d_M v_{i\alpha}.$$

Moreover, without a lack of generality we can assume that $w_i \in \Omega_M^{k-1}$. Therefore the statement follows by applying the principle of mathematical induction.

It is sufficient to check that (11) holds on elements of the form (10). Because of the graded Leibniz rule it is sufficient to check that $d_M^2(\mathcal{V}) = \{0\}$. However, this directly follows from (5) and from the definition of ∇ .

Finally, (i) follows from the graded Leibniz rule, (ii) and (iii), and from the fact that maps from $\mathfrak{fr}(P)$ act on elements from \mathcal{V} in the same way (fixed by τ). \square

In other words Ω_M , endowed with d_M , becomes a graded-differential $*$ -algebra generated by \mathcal{V} .

All basic structural elements of the conceptual framework of [3] are now in the game. Let us recall that a *preconnection* on P (relative to $\{\mathfrak{hor}_P, F^\wedge, \Omega_M\}$) is a first-order hermitian antiderivation $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ satisfying

$$(13) \quad F^\wedge D = (D \otimes \text{id}) F^\wedge$$

$$(14) \quad D \lrcorner \Omega_M = d_M$$

(according to (13) every D is reduced in Ω_M). Preconnections form a real affine space $\pi(P)$. Evidently, $\mathfrak{fr}(P) \subseteq \pi(P)$.

According to [3], for each $D \in \pi(P)$ and $E \in \overrightarrow{\pi}(P)$ (the vector space associated to $\pi(P)$) there exists the unique linear maps $\varrho_D^*, \chi_E^*: \mathcal{A} \rightarrow \mathfrak{hor}_P$ such that

$$(15) \quad D^2(\varphi) = - \sum_k \varphi_k \varrho_D^*(c_k),$$

$$(16) \quad E(\varphi) = -(-)^{\partial\varphi} \sum_k \varphi_k \chi_E^*(c_k)$$

for each $\varphi \in \mathfrak{hor}_P$, where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. The maps ϱ_D^* and χ_E^* determine, in a natural manner, a bicovariant first-order $*$ -calculus Ψ on G . This calculus is based (in the sense of [7]) on the right \mathcal{A} -ideal $\mathcal{R} \subseteq \ker(\epsilon)$ consisting of elements annihilated by all ϱ_D^* and χ_E^* . These maps can be therefore factorized through \mathcal{R} . In such a way we obtain maps $\varrho_D, \chi_E: \Psi_{inv} \rightarrow \mathfrak{hor}_P$, where $\Psi_{inv} = \ker(\epsilon)/\mathcal{R}$ is the space of left-invariant elements of Ψ . More precisely,

$$\begin{aligned} \varrho_D \pi &= \varrho_D^* \\ \chi_E \pi &= \chi_E^* \end{aligned}$$

where $\pi: \mathcal{A} \rightarrow \Psi_{inv}$ is the canonical projection map.

The following equalities hold

$$(17) \quad F^\wedge \varrho_D = (\varrho_D \otimes \text{id})\varpi$$

$$(18) \quad D\varrho_D = 0$$

$$(19) \quad F^\wedge \chi_E = (\chi_E \otimes \text{id})\varpi$$

where $\varpi: \Psi_{inv} \rightarrow \Psi_{inv} \otimes \mathcal{A}$ is the dualized (co)adjoint action, explicitly given by

$$\varpi\pi = (\pi \otimes \text{id})\text{ad},$$

and $\text{ad}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is the dualized adjoint action of G on itself. Further,

$$(20) \quad \chi_E(\vartheta)\varphi = (-)^{\partial\varphi} \sum_k \varphi_k \chi_E(\vartheta \circ c_k)$$

$$(21) \quad \varrho_D(\vartheta)\varphi = \sum_k \varphi_k \varrho_D(\vartheta \circ c_k)$$

for each $\vartheta \in \Psi_{inv}$ and $\varphi \in \mathfrak{hor}_P$, where \circ is the canonical right \mathcal{A} -module structure.

The map ϱ_D is called *the curvature* of D .

Every element $\varphi \in \mathfrak{hor}_P$ can be written in the form

$$\varphi = \sum_k b_k d_M w_k,$$

where $b_k \in \mathcal{B}$ and $w_k \in \Omega_M$. This implies that each $D \in \pi(P)$ is completely determined by its restriction $D|_{\mathcal{B}}$. Explicitly, this restriction is described by

$$(22) \quad D(b) = \sum_{i=1}^n Y_i(b)\theta_i,$$

where $Y_i: \mathcal{B} \rightarrow \mathcal{B}$ are hermitian derivations satisfying

$$(23) \quad Y_i|_{\mathcal{V}} = \partial_i$$

$$(24) \quad FY_i(b) = \sum_{kj} Y_j(b_k) \otimes c_k u_{ji}$$

for each $b \in \mathcal{B}$, where $\sum_k b_k \otimes c_k = F(b)$. In terms of derivations Y_i the action of D on special horizontal forms θ_j is given by

$$(25) \quad D(\theta_j) = \frac{1}{2} \sum_{\alpha kl} \left\{ Y_k(b_{j\alpha}) \partial_l(v_{j\alpha}) - Y_l(b_{j\alpha}) \partial_k(v_{j\alpha}) \right\} \theta_k \theta_l.$$

For every $D \in \pi(P)$ let $\Theta_D: \mathbb{C}^n \rightarrow \mathfrak{hor}_P$ be a linear map given by

$$(26) \quad \Theta_D(e_i) = \Theta_D^i = D(\theta_i).$$

Definition 2.3. The map Θ_D is called *the torsion* of D .

Lemma 2.2. *The map Θ_D is hermitian and satisfies*

$$(27) \quad F^\wedge \Theta_D^i = \sum_{j=1}^n \Theta_D^j \otimes u_{ji}$$

$$(28) \quad -D\Theta_D^i = \sum_{j=1}^n \theta_j \varrho_D^*(u_{ji})$$

for each $i \in \{1, \dots, n\}$.

Proof. The statement follows directly from properties (13) and (15) and from the definition of Θ_D and F^\wedge . \square

Identity (28) corresponds to the second Structure equation in classical differential geometry [5].

Lemma 2.3. *The following equivalence holds*

$$(29) \quad \Theta_D = 0 \iff D \in \mathfrak{fr}(P)$$

for each $D \in \pi(P)$.

Proof. Let us assume that $D \in \mathfrak{fr}(P)$. From (2) and (25) it follows that the torsion vanishes. Conversely, if $\Theta_D = 0$ then

$$0 = d_M^2(f) = D^2(f) = \frac{1}{2} \sum_{ij} [Y_i, Y_j](f) \theta_i \theta_j$$

for each $f \in \mathcal{V}$. In other words, $D \in \mathfrak{fr}(P)$. \square

Let us assume that the higher-order differential calculus on G is described by the universal envelope Ψ^\wedge of Ψ . Let Ω_P be the graded-differential $*$ -algebra canonically associated, in the sense of [3], to algebras \mathfrak{hor}_P and Ψ^\wedge , and to the system $\pi(P)$ of preconnections. The main property of this algebra is that each $D \in \pi(P)$ naturally induces a representation of the form

$$(30) \quad \Omega_P \leftrightarrow \mathfrak{vh}_P$$

where \mathfrak{vh}_P is a graded $*$ -algebra representing “vertically-horizontally” decomposed forms [2] on P . At the level of graded vector spaces, we have

$$\mathfrak{vh}_P = \mathfrak{hor}_P \otimes \Psi_{inv}^\wedge$$

The $*$ -algebra structure on \mathfrak{vh}_P is specified by

$$\begin{aligned} (\psi \otimes \vartheta)(\varphi \otimes \eta) &= \sum_k (-)^{\partial \vartheta \partial \varphi} \psi \varphi_k \otimes (\vartheta \circ c_k) \eta \\ (\varphi \otimes \vartheta)^* &= \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*). \end{aligned}$$

In terms of the identification (30) the differential structure on Ω_P is expressed via a D -dependent differential ∂_D on $\mathfrak{vh}(P)$, which is given by

$$\begin{aligned}\partial_D(\varphi) &= D(\varphi) + (-)^{\partial\varphi} \sum_k \varphi_k \pi(c_k) \\ \partial_D(\vartheta) &= \varrho_D(\vartheta) + d(\vartheta)\end{aligned}$$

where $\vartheta \in \Psi_{inv}$ and $d: \Psi_{inv}^\wedge \rightarrow \Psi_{inv}^\wedge$ is the corresponding differential (∂_D is extended on \mathfrak{vh}_P by the graded Leibniz rule).

Lemma 2.4. *As a differential algebra, Ω_P is generated by $\mathcal{B} = \Omega_P^0$.*

Proof. Because of (12), it is sufficient to check that elements of the form $\pi(a)$ belong to $\mathcal{B}\partial_D\{\mathcal{B}\}$. For a given $a \in \mathcal{A}$, let us choose elements $q_k, b_k \in \mathcal{B}$ such that $\sum_k q_k F(b_k) = 1 \otimes a$ (the group G acts freely on P). Then the following equality holds

$$\sum_k q_k \partial_D(b_k) = \sum_k q_k D(b_k) + \pi(a).$$

This implies (together with (12) and (22)) that $\pi(a)$ is expressible in the desired way. \square

The map F is uniquely extendible to the homomorphism $\widehat{F}: \Omega_P \rightarrow \Omega_P \widehat{\otimes} \Gamma^\wedge$ of graded differential $*$ -algebras (corresponding to the “pull back” map of differential forms).

There exists a natural bijective (affine) correspondence $D \leftrightarrow \omega$ between preconnections D and regular connections ω on P . In terms of this correspondence,

$$\begin{aligned}R_\omega &\leftrightarrow \varrho_D \\ D_\omega &\leftrightarrow D.\end{aligned}$$

It is also possible to construct differential structures on G and P starting from a restricted set of preconnections forming an affine subspace of $\mathfrak{fr}(P)$. In this case covariant derivatives of regular connections will (generally) form only an affine subspace of $\pi(P)$. In particular, every *single* element $\nabla \in \mathfrak{fr}(P)$ determines a differential calculus on P . Let us assume that Ψ is the minimal bicovariant first-order differential $*$ -calculus over G compatible with ∇ . Then the corresponding calculus on P will be based on a graded-differential $*$ -algebra $\Omega_P = (\mathfrak{vh}_P, \partial_\nabla)$. It is also possible to vary this theme, and to choose for Ψ an arbitrary (non-minimal) calculus satisfying the mentioned compatibility conditions.

Let $\pi(P)_\nabla \subseteq \pi(P)$ be the affine subspace consisting of preconnections interpretable as covariant derivatives of regular connections (relative to Ω_P constructed from $\{\nabla\}$).

A particularly interesting situation arises when ∇ is compatible with *the classical* differential calculus on G .

Definition 2.4. An element $\nabla \in \mathfrak{fr}(P)$ is called *classical* iff

$$\varrho_\nabla^*(ab) = \epsilon(a)\varrho_\nabla^*(b) + \varrho_\nabla^*(a)\epsilon(b)$$

for each $a, b \in \mathcal{A}$.

If ∇ is classical then it is possible to assume that Ψ is the classical differential calculus on G (hence $\Psi_{inv} = \text{lie}(G)^*$). In this particular case,

$$\begin{aligned}\varrho_D^*(ab) &= \epsilon(a)\varrho_D^*(b) + \varrho_D^*(a)\epsilon(b) \\ \chi_E^*(ab) &= \epsilon(a)\chi_E^*(b) + \chi_E^*(a)\epsilon(b) \\ \varrho_D(\vartheta)\varphi &= \varphi\varrho_D(\vartheta) \\ \chi_E(\vartheta)\varphi &= (-)^{\partial\varphi}\varphi\chi_D(\vartheta)\end{aligned}$$

for each $D \in \pi(P)_\nabla$ and $E \in \overline{\pi}^\nabla(P)_\nabla$ (the \circ -structure on Ψ_{inv} is trivialized, because $\vartheta \circ a = \epsilon(a)\vartheta$).

Lemma 2.5. *We have*

$$(31) \quad \pi(P)_\nabla \cap \mathfrak{fr}(P) = \{\nabla\}$$

for each $\nabla \in \mathfrak{fr}(P)$.

Proof. Let us consider elements $D \in \pi(P)_\nabla$ and $\nabla \in \mathfrak{fr}(P)$, and let us assume that

$$\nabla \leftrightarrow (X_1, \dots, X_n) \quad D \leftrightarrow (Y_1, \dots, Y_n).$$

Derivations $Z_i = X_i - Y_i$ possess the following properties

$$(32) \quad FZ_i(b) = \sum_{kj} Z_j(b_k) \otimes c_k u_{ji}$$

$$(33) \quad Z_i(f) = 0$$

for each $b \in \mathcal{B}$ and $f \in \mathcal{V}$. Applying results of [3] we conclude that there exist linear maps $\lambda_i: \Psi_{inv} \rightarrow \mathcal{B}$ such that

$$(34) \quad Z_i(b) = \sum_k b_k \lambda_i(\pi(c_k))$$

for each $b \in \mathcal{B}$ and $i \in \{1, \dots, n\}$. In particular,

$$(35) \quad Z_i \partial_j(f) = \sum_k \partial_k(f) \lambda_{ij}^k$$

where

$$(36) \quad \lambda_{ij}^k = \lambda_i(\pi(u_{kj})).$$

Let us assume that $\Theta_D = 0$. This is equivalent to

$$(37) \quad (Z_i \partial_j - Z_j \partial_i)(f) = 0$$

for each $f \in \mathcal{V}$ and $i, j \in \{1, \dots, n\}$. Identities (35)–(37), together with the completeness condition imply

$$(38) \quad \lambda_{ij}^k = \lambda_{ji}^k$$

On the other hand

$$(39) \quad \lambda_{ij}^k = -\lambda_{ik}^j$$

as easily follows from (36) and the hermicity of u_{ij} . It follows that $\lambda_{ij}^k = 0$, and hence $\lambda_i = 0$, because Ψ_{inv} is spanned by elements $\pi(u_{ij})$. Hence $D = \nabla$. \square

The above equivalence corresponds to the classical characterization of the Levi-Civita connection, as the unique (metric) connection with vanishing torsion. Conceptually, we followed a classical proof [5] of the uniqueness of the Levi-Civita connection.

3. EXAMPLES

3.1. The classical case

Let P be a classical principal $\mathrm{SO}(k)$ -bundle over a compact smooth n -dimensional manifold M (where $k \leq n$) and let $\tau = (\partial_1, \dots, \partial_k)$ be a frame structure on P (relative to the standard representation u of $\mathrm{SO}(k)$ in \mathbb{C}^k). Then every point $p \in P$ naturally determines a k -tuple (ξ_1, \dots, ξ_k) on tangent vectors on M in the point $x = \pi_M(p)$ as follows

$$(40) \quad \xi_i(f) = [\partial_i(f)](p).$$

Here $\pi_M: P \rightarrow M$ is the projection map.

From the transformation property (1), it follows that the space $\Sigma_x \subseteq T_x(M)$ spanned by (ξ_1, \dots, ξ_k) is independent of the choice of the point $p \in \pi_M^{-1}(x)$. On the other hand, completeness condition (2) implies that (ξ_1, \dots, ξ_k) are linearly independent vectors.

In such a way an oriented k -dimensional subbundle Σ of $T(M)$ is constructed. Fibers $(\Sigma_x)_{x \in M}$ possess a natural Euclidean structure defined by requiring that (ξ_1, \dots, ξ_k) are orthonormal vectors. In classical terms, P is identifiable with the bundle of oriented orthonormal frames of Σ .

Let us assume that τ is integrable. This implies that the space of smooth sections of Σ is closed with respect to the commutator of vector fields. In other words, Σ is integrable (according to Frobenius theorem).

Let $N \subseteq M$ be an arbitrary leaf of the foliation Σ , and P_N the portion of P over N . This bundle coincides, in a natural manner, with the bundle of oriented orthonormal frames of N . It is invariant under the action of fields X_i . Restrictions of X_i on P_N determine the standard Levi-Civita connection.

Bundles P_N determine a foliation Σ^* of P . The elements of Ω_P are naturally interpretable as Σ^* -differential forms on P . In this picture the algebra Ω_M consists of Σ -differential forms on M .

In particular, the case $k = n$ is equivalent to the classical oriented Riemannian manifold structure on M , so that P becomes the corresponding bundle of oriented orthonormal frames.

3.2. A Framed Quantum $\mathrm{SO}(2)$ Bundle

Let us assume that \mathcal{V} is endowed with a $*$ -automorphism $\gamma: \mathcal{V} \rightarrow \mathcal{V}$. Let us assume that $G = \mathrm{SO}(2)$. The Hopf $*$ -algebra \mathcal{A} of polynomial functions on G is generated by a unitary element $U = \cos + i \sin$ (and $\{\cos, \sin\}$ are understood as

functions on G). We have $\phi(U) = U \otimes U$. The formulas

$$(41) \quad (f \otimes U^m)(g \otimes U^n) = f\gamma^m(g) \otimes U^{m+n}$$

$$(42) \quad (f \otimes U^m)^* = \gamma^{-m}(F^\wedge) \otimes U^{-m}$$

(where $n, m \in \mathbb{Z}$) define a $*$ -algebra structure on the vector space $\mathcal{B} = \mathcal{V} \otimes \mathcal{A}$. Let $i: \mathcal{V} \rightarrow \mathcal{B}$ be the canonical inclusion map. The formula

$$F(f \otimes U^m) = f \otimes U^m \otimes U^m$$

defines an action of G by $*$ -automorphisms of \mathcal{B} , so that $P = (\mathcal{B}, i, F)$ is a (quantum) principal G -bundle over M .

Let us define derivations $X_\pm: \mathcal{B} \rightarrow \mathcal{B}$ by

$$(43) \quad X_+(b) = (\alpha \otimes U)b - b(\alpha \otimes U)$$

$$(44) \quad X_-(b) = (\beta \otimes \bar{U})b - b(\beta \otimes \bar{U})$$

where $\alpha, \beta \in \mathcal{V}$ are such that $\beta = -\gamma^{-1}(\alpha^*)$. We have

$$[X_+, X_-](b) = vb - bv$$

where $v = \alpha\gamma(\beta) - \beta\gamma^{-1}(\alpha)$. Let $X_1, X_2: \mathcal{B} \rightarrow \mathcal{B}$ be (hermitian) derivations given by $X_\pm = X_1 \mp iX_2$. Let $\partial_1, \partial_2 = X_1, X_2|_{\mathcal{V}}$ be the corresponding restrictions. If the completeness condition (2) holds then $\tau = (\partial_1, \partial_2)$ is a frame structure on P , relative to the standard representation

$$u = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$$

of $\text{SO}(2)$ in \mathbb{C}^2 . If v is a central element of \mathcal{V} then τ is integrable, and we can write $\hat{\tau} = (X_1, X_2)$. The corresponding curvature is given by

$$(45) \quad \varrho_\nabla^*(U^m) = \frac{1}{4i}(v - \gamma^{-m}(v))\theta_1\theta_2.$$

In general, such frame structures induce nonstandard differential calculi on G . Let us assume that v is non-trivial, and that $\gamma(v) = tv$, for some $t \in \mathbb{R} \setminus \{-1, 0, 1\}$. This naturally induces a 1-dimensional calculus on G .

The space Ψ_{inv} is spanned by $\zeta = \pi(U - \bar{U})$. The corresponding ideal \mathcal{R} is generated by $tU + \bar{U} - (1+t)1$. The \circ -structure on Ψ_{inv} is specified by $\zeta \circ U^m = t^{-m}\zeta$.

3.3. Free Actions of Simple Lie Groups on Quantum Spaces

Let us assume that a compact simple Lie group H acts freely on a quantum space P (determined by a $*$ -algebra \mathcal{B}). Let G be a compact subgroup of H . Let $F^*: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}^*$ be the dualized action of H on P (where \mathcal{A}^* is the $*$ -algebra of polynomial functions on H) and let $F = (\text{id} \otimes j)F^*$ be the restriction of the action of H on G . Here $j: \mathcal{A}^* \rightarrow \mathcal{A}$ is the restriction map. In what follows the entities endowed with \star will refer to H . Both groups will be endowed with standard differential structures.

The triplet $P = (\mathcal{B}, i, F)$ is a quantum principal G -bundle over M , where M is the quantum space based on the F -fixed-point $*$ -subalgebra \mathcal{V} , and $i: \mathcal{V} \hookrightarrow \mathcal{B}$ is the inclusion map.

Let \mathfrak{g}^* be the (complex) Lie algebra of H . Let u be the adjoint representation of G in the space $\mathfrak{g}^\perp \subseteq \mathfrak{g}^*$. Here, $\mathfrak{g} \subseteq \mathfrak{g}^*$ is the Lie algebra of G , and it is assumed that \mathfrak{g}^* is endowed with the (positive) Killing scalar product. Let us assume that the kernel of u is discrete. Furter, let us assume that \mathfrak{g}^\perp is identified with \mathbb{C}^n , with the help of a real orthonormal basis (ξ_1, \dots, ξ_n) in \mathfrak{g}^\perp .

For each $i \in \{1, \dots, n\}$ let $X_i = (\text{id} \otimes \xi_i \pi^*) F^*$ be the hermitian derivation on \mathcal{B} corresponding to ξ_i (here we have identified $\mathfrak{g}^* = (\Psi_{inv}^*)^*$)

Let $\partial_i: \mathcal{V} \rightarrow \mathcal{B}$ be the restrictions of X_i on \mathcal{V} .

Lemma 3.1. *Under the above assumptions $\tau = (\partial_1, \dots, \partial_n)$ is a frame structure on P . This structure is integrable if*

$$(46) \quad [\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}.$$

In this case

$$(47) \quad \hat{\tau} = (X_1, \dots, X_n).$$

Proof. A direct computation gives

$$\begin{aligned} FX_i(b) &= \sum_k F(b_k \xi_i \pi^*(d_k)) \\ &= \sum_k b_k \xi_i \pi^*(d_k^{(2)}) \otimes j \left[d_k^{(1)} \kappa^*(d_k^{(3)}) d_k^{(4)} \right] = \sum_{jk} X_j(b_k) \otimes u_{ji} j(d_k). \end{aligned}$$

Here, $F^*(b) = \sum_k b_k \otimes d_k$ and we have applied the identity

$$\xi_i \pi^*(a^{(2)}) \otimes j(a^{(1)} \kappa^*(a^{(3)})) = \sum_j \xi_j \pi^*(a) \otimes u_{ji}.$$

Hence, derivations X_i transform in the appropriate way.

Let us prove the completeness condition. It is sufficient to prove that for each $a \in \mathcal{A}^*$, invariant under the right action of G on H , there exist elements $b_k \in \mathcal{B}$ and $v_k \in \mathcal{V}$ such that

$$(48) \quad \sum_k b_k F^*(v_k) = 1 \otimes a.$$

Indeed, for a given G -invariant element $a \in \mathcal{A}^*$ there exist elements $q_k, b_k \in \mathcal{B}$ such that

$$\sum_k b_k F^*(q_k) = 1 \otimes a$$

(this is the place where the freeness assumption enters the game). This implies

$$\sum_{kl} b_k q_{kl} \otimes a_{kl}^{(1)} h j(a_{kl}^{(2)}) = 1 \otimes a$$

where $F^*(q_k) = \sum_l q_{kl} \otimes a_{kl}$ and $h: \mathcal{A} \rightarrow \mathbb{C}$ is the Haar measure on G . If we define

$$v_k = \sum_l q_{kl} h j(a_{kl}),$$

then $v_k \in \mathcal{V}$ and (48) holds.

Finally, let us assume that (46) holds. Then $[X_i, X_j](v) = 0$ for each $v \in \mathcal{V}$ and $i, j \in \{1, \dots, n\}$, according to the definition of \mathcal{V} . Hence, τ is integrable and (47) holds. \square

The corresponding graded-differential $*$ -algebra Ω_P can be naturally realized as

$$\Omega_P = [\Psi_{iv}^*]^\wedge \otimes \mathcal{B},$$

in other words, \mathcal{B} -valued forms on \mathfrak{g}^* (with the standard algebraic structure).

The curvature map of the Levi-Civita connection ∇ is given by

$$\varrho_\nabla^*(a) = -\frac{1}{2} \sum_{ij} [\xi_i, \xi_j] \pi(a) \theta_i \theta_j.$$

4. CONCLUDING REMARKS

Quantum counterparts of various important differential-geometric structures can be introduced in the framework of the concept of the frame structure. In particular, frame structures on quantum $\mathrm{SO}(n)$ -bundles provide a natural framework for a noncommutative-geometric version of (oriented) Riemannian geometry (the Xodge $*$ -operator and the Laplace operator, for example).

All essential elements of the algebraic structure appearing in the theory of Kahler manifolds are preserved in its noncommutative-geometric version, dealing with framed quantum $\mathrm{U}(n)$ -bundles. In particular, quantum spaces M considered in the previous section can be naturally endowed with Kahler manifold structures, in accordance with the analogy $M \leftrightarrow \mathrm{CP}(n)$.

In this paper, we have assumed that the structure quantum group is compact. This assumption is not essential. The whole formalism can be directly incorporated in a more general conceptual framework, including non-compact structure groups and non-unitary representations u . However in this case (as in classical geometry) ∇ is generally not uniquely determined by its class $\pi(P)_\nabla$, even if the calculus on the structure group is classical.

In terms of the constructed differential calculus on P , frame structures are completely represented by n -tuples $\theta = (\theta_1, \dots, \theta_n)$ of special horizontal 1-forms θ_i (counterparts of classical frame forms [5]). These forms transform covariantly, according to the representation u .

In this sense, integrable frame structures can be viewed as a special case of frame structures introduced in [4]–Appendix B.

The presented formalism admits a natural generalization to the fully quantum context (in which the structure group G is a quantum object). The only essentially new phenomena naturally appearing in the game is the presence of a non-trivial right \mathcal{A} -module structure \circ on the representation space \mathbb{C}^n , which is compatible with the right comodule structure u in such a way that \mathbb{C}^n together with u and \circ becomes a “left-invariant part” of a bicovariant [7] $*$ -bimodule Γ over G . This requires the following compatibility condition

$$u(e_i \circ a) = \sum_j (e_j \circ a^{(2)}) \otimes \kappa(a^{(1)}) u_{ji} a^{(3)}$$

for each $i \in \{1, \dots, n\}$ and $a \in \mathcal{A}$.

The external algebra \mathbb{C}_n^\wedge should be replaced by the quantum external algebra, associated to the “left-invariant part” $\sigma: \mathbb{C}^n \otimes \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n$ of the corresponding canonical flip-over [7] operator (or its scalar multiple). Also, the construction of the horizontal algebra incorporates elements of the bicovariant bimodule structure. Maps X_i are not derivations, although their restrictions $\partial_i = X_i|_{\mathcal{V}}$ still satisfy the Leibniz rule.

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